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Vector optimization problems with nonconvex preferences

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Abstract In this paper, some vector optimization problems are considered where pseudo-ordering relations are determined by nonconvex cones in Banach spaces. We give some characterizations of solution sets for vector complementarity problems and vector variational inequalities. When the nonconvex cone is the union of some convex cones, it is shown that the solution set of these problems is either an intersection or an union of the solution sets of all subproblems corresponding to each of these convex cones depending on whether these problems are defined by the nonconvex cone itself or its complement. Moreover, some relations of vector complementarity problems, vector variational inequalities, and minimal element problems are also given.

Keywords Vector complementarity problem · vector variational inequality · vector optimization problem · nonconvex cone

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While this paper was being revised in September 2006, Professor Alex Rubinov (the second author of the paper) left us due to the illness. This is a very sad news to us. We dedicate this paper to the memory of Professor Rubinov as a mathematician and truly friend.

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1 Introduction

Vector variational inequality was first introduced and studied by Giannessi [5] in the setting of finite-dimensional Euclidean spaces. Since then, existence results and duality for vector variational inequalities and vector complementarity problems have been studied by many authors (see, for e.g., [2,6,7,10] and the references therein).

Chen and Yang [3] discussed equivalence relations among a vector complementarity problem, a vector variational inequality problem, a vector extremum problem, a weak minimal element problem, and a vector unilateral minimization problem in Banach spaces. We refer to [1,4,8,9,11,14] for some related works.

Recently, Rubinov and Gasimov [13] considered a vector optimization problem with preferences that are not necessarily a pre-order relation. They studied a class of preferences that are defined by means of so-called strongly star-shaped conic sets in a Banach space X. The simplest example of a strongly star-shaped conic set is the union of a finite number of convex and closed cones with the intersection having a nonempty interior. Such a relation, determined by a nonconvex cone, is not transitive. Thus, there might be some difficulties to study corresponding vector optimization problems. However, Rubinov and Gasimov [13] suggested certain classes of functions that provide scalarization of the relations. Using this class they constructed scalar optimization problems such that weakly minimal points, minimal points and proper minimal points can be completely described as solutions of these problems.

This paper aims to understand the solution structure for vector optimization problems where the ordering cone is not convex. These results may be useful in the design of optimal algorithms to find the whole solution set of vector optimization problems with a nonconvex ordering cone. We introduce vector complementarity problems, vector variational inequalities, and vector optimization problems where relations are determined by a nonconvex cone in Banach spaces. We give some characterization results of solution sets for vector complementarity problems and vector variational inequalities. More specifically, when the nonconvex ordering cone is defined as the union of a set of closed and convex cones, the solution sets of the above problems can be represented in terms of the intersection or union of the solution sets of the corresponding subproblems defined by each closed and convex cone. Some simple examples are given to illustrate these relationships.

We also give some relations of vector complementarity problems, vector variational inequalities, and minimal element problems.

2 Vector optimization problems

In this section, we give some results concerned with relations of solution sets for (mild) strong vector complementarity problems, (mild) strong vector variational inequalities, and (mild) strong vector optimization problems.

Let X be a Banach space with a dual space X^* and A be a subset of X. The topological interior of a subset A in X is denoted by $intA$. A nonempty subset C in X is called a cone if $\lambda C \subset C$ for any $\lambda > 0$. A cone C is called a convex cone if $C + C = C$. A subset C is called a pointed cone if C is a cone and $C \cap (-C) = \{0\}$.

Let (X, C) be an ordered Banach space with C being convex and $intC \neq \emptyset$.

Let P be a cone of a Banach space Y and let $\mathcal{C}(P)$ denote the complement of P. Since P is a cone, we know that $\mathcal{C}(P)$ is also a cone.

Let P be a cone of a Banach space Y. We define the following ordering relation: for any $y_1, y_2 \in Y$,

$$y_1 \geq_P y_2 \quad \text{if and only if} \quad y_1 - y_2 \in P.$$

Note that the ordering $\geq_P$ may not be transitive. In sequel, P may be one of the following: P itself, $\mathcal{C}(intP)$ and $\mathcal{C}(P\backslash\{0\})$.

Let Y be a Banach space and P be a closed and pointed cone in Y with $intP \neq \emptyset$. Let $L(X, Y)$ be the space of all continuous linear maps from X to Y and $T: X \to L(X, Y)$. We denote the value of $l \in L(X, Y)$ at $x \in X$ by (l, x). Consider the following problems:

Strong Vector Complementarity Problem (SVCP): find $x \in C$ such that

$$(Tx, x) = 0, \quad (Tx, y) \geq_P 0, \quad \forall y \in C;$$

Mild Vector Complementarity Problem (MVCP): find $x \in C$ such that

$$(Tx, x) = 0, \quad (Tx, y) \leq_{\mathcal{C}(P\backslash\{0\})} 0, \quad \forall y \in C;$$

Positive Vector Complementarity Problem (PVCP): find $x \in C$ such that

$$(Tx, x) \geq_{\mathcal{C}(intP)} 0, \quad (Tx, y) \geq_P 0, \quad \forall y \in C;$$

Strong Vector Variational Inequality (SVVI): find $x \in C$ such that

$$(Tx, y - x) \geq_P 0, \quad \forall y \in C;$$

Mild Vector Variational Inequality (MVVI): find $x \in C$ such that

$$(Tx, y - x) \leq_{\mathcal{C}(P\backslash\{0\})} 0, \quad \forall y \in C;$$

and

Strong Minty Vector Variational Inequality (SMVVI): find $x \in C$ such that

$$(Ty, y - x) \geq_P 0, \quad \forall y \in C.$$

We would like to point out that the most of the above problems have been introduced and studied by several authors when the cone P is convex (see, e.g., [2,3,6,7,9]).

2.1 P is a general nonconvex cone

We need the following notions.

Definition 2.1 A mapping $T: X \to L(X, Y)$ is said to be pseudomonotone with respect to P if, for any $x, y \in X$,

$$(Tx, y - x) \geq_P 0 \Longrightarrow (Ty, y - x) \geq_P 0.$$

Example 2.1 Let $X = R$, $C = [0, +\infty)$, $Y = R^2$ and

$$P = \left\{(x, y) : x \geq 0, 0 \leq y \leq \frac{x}{2}\right\} \cup \left\{(x, y) : y \geq 0, 0 \leq x \leq \frac{y}{2}\right\}.$$

Then P is a nonconvex cone. Let $T: X \to L(X, Y)$ be defined as follows:

$$(Tx, z) = (x^2 + 1)(2z, z), \quad \forall x, z \in X.$$

Then it is easy to verify that T is pseudomonotone.

Example 2.2 Let $X = Y = L_p(S, \Sigma, \mu)$ and $A \in L(L_p, L_p)$, where (S, Σ, μ) is a measure space, $p \geq 1$, and $L_p(S, \Sigma, \mu)$ is the class of all measurable functions f such that $|f|^p$ is μ-integrable. For $x \in L_p$ define $e(x) = \{s \in S : (Ax)(s) < 0\}$. Let $P = \{x \in L_p : \mu(e(x)) > 0\}$. Then it is easy to check that P is a cone. Let $T: L_p \to L(L_p, L_p)$ be defined by

$$(Tx, y)(s) = \begin{cases} (Ay)(s), & \text{if } s \in e(x), \\ 0, & \text{if } s \notin e(x). \end{cases}$$

Then

$$(Tx, y - x)(s) = \begin{cases} A(y - x)(s), & \text{if } s \in e(x), \\ 0, & \text{if } s \notin e(x) \end{cases}$$

and

$$\begin{aligned} e(Tx, y - x) &= \{s \in S : (Tx, y - x) < 0\} \\ &= \{s \in S : (Ax)(s) < 0, A(y - x)(s) < 0\}. \end{aligned}$$

Now we check that $e(Ty, y - x) \supset e(Tx, y - x)$. Indeed, letting $s \in e(Tx, y - x)$, then $(Ax)(s) < 0$, $(Ay)(s) - (Ax)(s) < 0$. It follows that $(Ay)(s) < (Ax)(s) < 0$ and so

$$s \in e(Ty, y - x) = \{s \in S : (Ay)(s) < 0, (Ay)(s) < (Ax)(s)\}.$$

Thus, if $(Tx, y - x) \in P$ then $\mu(e(Tx, y - x)) > 0$, so $\mu(e(Ty, y - x)) > 0$. This implies that the mapping T is pseudomonotone.

Definition 2.2 A mapping $T: X \to L(X, Y)$ is said to be hemicontinuous if, for any given $x, y \in X$, the mapping $t \mapsto (T(x + t(y - x)), y - x)$ is continuous at 0^+.

Let S^P_{SVCP}, S^P_{SMVVI}, S^P_{SVVI}, S^P_{MVCP}, and S^P_{MVVI} denote the solution sets of (SVCP), (SMVVI), (SVVI), (MVCP), and (MVVI), respectively.

Theorem 2.1 *For any $T: X \to L(X, Y)$, we have the following results:*

(1) $S^P_{\mathrm{SVVI}} = S^P_{\mathrm{SVCP}}$;
(2) *If T is hemicontinuous and pseudomonotone, then $S^P_{\mathrm{SMVVI}} = S^P_{\mathrm{SVVI}}$.*

Proof (1) Letting $x \in S^P_{\mathrm{SVCP}}$, then $x \in C$ and

$$(Tx, x) = 0, \quad (Tx, y) \geq_P 0, \quad \forall y \in C.$$

Thus, for any $y \in C$,

$$\begin{aligned} (Tx, y - x) &= (Tx, y) - (Tx, x) \\ &= (Tx, y) - 0 \\ &\geq{}_P 0 \end{aligned}$$

and so $x \in S^P_{\mathrm{SVVI}}$. Conversely, suppose that $x \in S^P_{\mathrm{SVVI}}$. Then

$$(Tx, y - x) \geq_P 0, \quad \forall y \in C.$$

Since C is a cone, putting $y = 2x$ and $y = 0$ in the above inequality, we have

$$(Tx, x) \geq_P 0, \quad -(Tx, x) \geq_P 0$$

and so

$$(Tx, x) \in P \cap (-P).$$

Since $P \cap (-P) = \{0\}$, we know that $(Tx, x) = 0$. Furthermore, for any $y \in C$,

$$(Tx, y) = (Tx, y - x) + (Tx, x) \geq_P 0.$$

It follows that $x \in S^P_{\text{SVCP}}$ and so $S^P_{\text{SVVI}} = S^P_{\text{SVCP}}$.

(2) Suppose that $x \in S^P_{\text{SVVI}}$. Then

$$(Tx, y - x) \geq_P 0, \quad \forall y \in C.$$

Since T is pseudomonotone,

$$(Ty, y - x) \geq_P 0, \quad \forall y \in C$$

and so $x \in S^P_{\text{SMVVI}}$. Conversely, letting $x \in S^P_{\text{SMVVI}}$, we have

$$(Ty, y - x) \geq_P 0, \quad \forall y \in C.$$

For any $y \in C$, let $z = ty + (1-t)x$. Then $z \in C$ for $t \in (0, 1)$. Substituting $z = ty + (1-t)x$ into the above inequality, we have

$$t(T(x + t(y - x)), y - x) \geq_P 0, \quad \forall y \in C.$$

Since P is a cone, it follows that

$$(T(x + t(y - x)), y - x) \geq_P 0, \quad \forall y \in C.$$

The hemicontinuity of T implies that

$$(Tx, y - x) \geq_P 0, \quad \forall y \in C$$

and so $x \in S^P_{\text{SVVI}}$. This completes the proof. □

It follows from Theorem 2.1 (1) that the following result holds.

Theorem 2.2 *For any $T: X \to L(X, Y)$, we have $S^P_{\text{MVCP}} \subset S^P_{\text{MVVI}}$.*

Let A be a nonempty subset of Y. $a \in A$ is said to be a strongly (or an ideal) minimal point of the set A with respect to P if $a \leq_P y$ for all $y \in A$. $a \in A$ is said to be a mildly minimal point of the set A with respect to P if $y \leq_{C(P \setminus \{0\})} a$ for all $y \in A$. We denote by $\text{Min}_P A$ and $\text{Min}_{C(P \setminus \{0\})} A$ the set of all strongly minimal points of A and the set of all mildly minimal points of A, respectively.

Let $T: X \to L(X, Y)$ be a mapping. Define the feasible sets $\mathcal{F}_s$ and $\mathcal{F}_m$ associated with T by

$$\mathcal{F}_s = \{x \in X : \; x \in C, \;\; (Tx, y) \geq_P 0, \quad \forall y \in C\}$$

and

$$\mathcal{F}_m = \{x \in X : \; x \in C, \;\; (Tx, y) \leq_{C(P \setminus \{0\})} 0, \quad \forall y \in C\},$$

respectively.

Let $f(x) = (Tx, x)$ for all $x \in C$. We now consider the following problems:

Strong Vector Optimization Problem (SVOP):

$$\text{Min}_P f(x) \qquad \text{subject to} \quad x \in \mathcal{F}_s$$

and
Mild Vector Optimization Problem (MVOP):

$$\mathrm{Min}_{C(P\setminus\{0\})} f(x) \qquad \text{subject to} \quad x \in \mathcal{F}_m.$$

A point x is called a strongly minimal solution of SVOP (resp., a mildly minimal solution of MVOP) if $f(x)$ is a strongly minimal point of SVOP (resp., a mildly minimal point of MVOP), i.e., $f(x) \in \mathrm{Min}_P f(\mathcal{F}_s)$ (resp., $f(x) \in \mathrm{Min}_{C(P\setminus\{0\})} f(\mathcal{F}_m)$). We denote the set of all strongly minimal solutions of SVOP (resp., mildly minimal solutions of MVOP) by E_s (resp., E_m) and the set of all strongly minimal points of SVOP(resp., mildly minimal points of MVOP) by H_s(resp., H_m). Then $f(E_s) = H_s$ (resp., $f(E_m) = H_m$).

Theorem 2.3 *Suppose that $f(E_s) \neq \emptyset$. Then the following conclusions hold:*

(1) *if there exists $x \in E_s$ such that $f(x) = 0$, then the SVCP is solvable;*
(2) *if there exists $x \in E_s$ such that $f(x) \geq_{C(intP)} 0$, then the PVCP is solvable.*

Proof It is easy to see that (1) is true. Now we prove that (2) holds. Let $x \in E_s$ and $f(x) \geq_{C(intP)} 0$. Then $x \in C$ and

$$(Tx, x) = f(x) \geq_{C(intP)} 0, \quad (Tx, y) \geq_P 0, \quad \forall y \in C.$$

It follows that x is a solution of PVCP. This completes the proof.
Similarly, we have the following result. □

Theorem 2.4 *Suppose that $f(E_m) \neq \emptyset$. If there exists $x \in E_m$ such that $f(x) = 0$, then the mild strong vector complementarity problem (MVCP) is solvable.*

We now consider the following problems:
The SVOP$_l$: for a given $l \in L(X, Y)$, finding $x \in \mathcal{F}_s$ such that $l(x) \in \mathrm{Min}_P l(\mathcal{F}_s)$;
The strongly minimal element problem (SMEP): finding $x \in \mathcal{F}_s$ such that $x \in \mathrm{Min}_C \mathcal{F}_s$;
The strong vector unilateral optimization problem (SVUOP): finding $x \in C$ such that $f(x) \in \mathrm{Min}_P f(C)$;
The MVOP$_l$: for a given $l \in L(X, Y)$, finding $x \in \mathcal{F}_m$ such that $l(x) \in \mathrm{Min}_{C(P\setminus\{0\})} l(\mathcal{F}_m)$;
The mild minimal element problem (MMEP): finding $x \in \mathcal{F}_m$ such that $x \in \mathrm{Min}_{C(P\setminus\{0\})} \mathcal{F}_m$.
Let X and Y be two Banach spaces. A map $f: X \to Y$ is Frechet differentiable at $x_0 \in X$ if there exists a linear bounded operator $Df(x_0)$ such that

$$\lim_{x \to 0} \|f(x_0 + x) - f(x_0) - (Df(x_0), x)\| / \|x\| = 0.$$

In this case, $Df(x_0)$ is said to be the Frechet derivative of f at x_0. The map f is said to be Frechet differentiable on X if f is Frechet differentiable at each point of X.

Theorem 2.5 *Let $T = Df$ be the Frechet derivative of an operator $f: X \to Y$. Then x solves (SVUOP) implies that x solves (SVVI).*

Proof Let x be a solution of (SVUOP). Then $x \in C$ and $f(x) \in \mathrm{Min}_P f(C)$, i.e., $f(x) \leq_P f(y)$ for all $y \in C$. Since, C is a convex cone,

$$f(x) \leq_P f(x + t(w - x)), \quad 0 < t < 1, \ w \in C.$$

It follows that

$$\frac{1}{t}[f(x+t(w-x))-f(x)] \geq_P 0.$$

Since f is Frechet differentiable on X and P is closed, letting $t \to 0^+$, we get

$$(Df(x), w-x) \geq_P 0, \quad \forall w \in C,$$

which is (SVVI). This completes the proof. □

Definition 2.3 A linear operator $l: X \to Y$ is called positive with respect to C and P if, for any $x, y \in X$,

$$x \geq_C y \Longrightarrow l(x) \geq_P l(y).$$

Example 2.3 Let $X = R = (-\infty, \infty)$, $C = [0, +\infty)$, $Y = R^2$ and

$$P = \left\{(x,y) : x \geq 0, 0 \leq y \leq \frac{x}{2}\right\} \cup \left\{(x,y) : y \geq 0, 0 \leq x \leq \frac{y}{2}\right\}.$$

Then P is a nonconvex cone. Let $l: X \to Y$ be defined as follows:

$$l(x) = \left(x, \frac{x}{4}\right), \quad \forall x \in X.$$

Then it is easy to verify that l is positive with respect to C and P.

Theorem 2.6 *Let l be a linear operator such that l is positive with respect to C and P. Then x solves (SMEP) implies that x solves $(SVOP)_l$.*

Proof Let x be a solution of SMEP. Then $x \in \mathcal{F}_s$ and $x \leq_C y$ for all $y \in \mathcal{F}_s$, where

$$\mathcal{F}_s = \{x \in X : \ x \in C, \ (Tx, y) \geq_P 0, \quad \forall y \in C\}.$$

For any $z \in \mathcal{F}_s$, we know that $x \leq_C z$. Since l is a positive linear operator, it follows that $l(x) \leq_P l(z)$ and so

$$l(x) \in \mathrm{Min}_P l(\mathcal{F}_s),$$

which is $(\mathrm{SVOP})_l$. This completes the proof. □

Similarly, we have the following result.

Theorem 2.7 *Let l be a linear operator. If l is positive with respect to $\mathcal{C}(C\backslash\{0\})$ and $\mathcal{C}(P\backslash\{0\})$, then x solves (MMEP) implies that x solves $(MVOP)_l$.*

2.2 P is a union of convex cones

We now consider the special case of P, that is, P is the union of convex cones.

Suppose that $P = \cup_{i \in I} P_i$, where I is an index set and P_i is convex, closed, and pointed cones. Then it is clear that P may be not convex. We now give some examples of nonconvex cones as follows.

Example 2.4 Let $C(Q)$ denote the space of all continuous functions on Q, where Q is compact. Let $E \subset Q$ be closed and

$$P = \left\{x \in C(Q) : \max_{t \in E} x(t) \geq 0\right\}.$$

Then $P = \bigcup_{t\in E} P_t$, where $P_t = \{x \in C(Q) : x(t) \geq 0\}$ is the half-space and thus a convex cone, so P is the union of convex cones. It is easy to check that P is a nonconvex cone.

Example 2.5 Let Q, $C(Q)$, and E be the same as in Example 2.4. Let

$$P' = \left\{x \in C(Q) : x(t) \leq 0, \quad \forall t \in Q \quad \text{and} \quad \max_{t\in E} x(t) = 0\right\}.$$

Then $P' = \bigcup_{T\in E} P'_\tau$, where

$$P'_\tau = \{x \in C(Q) : x(t) \leq 0, \quad \forall t \in Q \quad \text{and} \quad x(\tau) = 0\}.$$

It is easy to see that P' is a nonconvex cone.

Proposition 2.1 *Let $T: X \to L(X, Y)$. Suppose that $P = \bigcup_{i\in I} P_i$, where P_i is a convex cone in Y for $i \in I$. If T is pseudomonotone with respect to each P_i, $i \in I$, then T is pseudomonotone with respect to P.*

Proof Suppose that $(Tx, y - x) \geq_P 0$. Since $P = \bigcup_{i\in I} P_i$, there exists $i \in I$ such that $(Tx, y-x) \geq_{P_i} 0$. Since T is pseudomonotone with respect to each P_i, $(Ty, y-x) \geq_{P_i} 0$. Since $P = \bigcup_{i\in I} P_i$, we have $(Ty, y - x) \geq_P 0$. Therefore, T is pseudomonotone with respect to P. This completes the proof. □

Theorem 2.8 *Let $T: X \to L(X, Y)$ and $P = \bigcup_{i\in I} P_i$, where P_i is a closed, pointed, and convex cone in Y for $i \in I$. Then*

(1) $S^P_{\text{SVCP}} \supset \bigcup_{i\in I} S^{P_i}_{\text{SVCP}}$, *where $S^{P_i}_{\text{SVCP}}$ $(i \in I)$ denotes the solution set of the following strong vector complementarity problem: find $x \in C$ such that*

$$(Tx, x) = 0, \quad (Tx, y) \geq_{P_i} 0, \quad \forall y \in C;$$

(2) $S^P_{\text{PVCP}} \subset \bigcap_{i\in I} S^{P_i}_{\text{PVCP}}$, *where $S^{P_i}_{\text{PVCP}}$ $(i \in I)$ denotes the solution set of the following positive vector complementarity problem: find $x \in C$ such that*

$$(Tx, x) \geq_{\mathcal{C}(intP_i)} 0, \quad (Tx, y) \geq_P 0, \quad \forall y \in C;$$

(3) $S^P_{\text{MVCP}} = \bigcap_{i\in I} S^{P_i}_{\text{MVCP}}$, *where $S^{P_i}_{\text{MVCP}}$ $(i \in I)$ denotes the solution set of the following mild vector complementarity problem: find $x \in C$ such that*

$$(Tx, x) = 0, \quad (Tx, y) \leq_{\mathcal{C}(P_i\setminus\{0\})} 0, \quad \forall y \in C.$$

Proof (1) Let $x \in \cup_{i\in I} S^{P_i}_{\text{SVCP}}$. Then there exists $i \in I$ such that $x \in S^{P_i}_{\text{SVCP}}$. Thus, $x \in C$, $(Tx, x) = 0$, and

$$(Tx, y) \geq_{P_i} 0, \quad \forall y \in C.$$

This implies that $(Tx, y) \in P_i$ for all $y \in C$ and so $(Tx, y) \in P$ for all $y \in C$. It follows that $x \in S^P_{\text{SVCP}}$.

(2) Suppose that $x \in S^P_{\text{PVCP}}$. Then $x \in C$ and

$$(Tx, x) \geq_{\mathcal{C}(intP)} 0, \quad (Tx, y) \geq_P 0, \quad \forall y \in C.$$

This implies that $(Tx, x) \in \mathcal{C}(intP)$ and $(Tx, y) \in P$ for all $y \in C$. Since $P = \cup_{i\in I} P_i$, we know that

$$\bigcup_{i\in I} intP_i \subset intP.$$

It follows that $(Tx, x) \in \mathcal{C}(intP_i)$ for all $i \in I$. Thus, $x \in \cap_{i\in I} S^{P_i}_{\mathrm{PVCP}}$.
(3) Let $x \in S^{P}_{\mathrm{MVCP}}$. Then $x \in C$,

$$(Tx, x) = 0, \quad (Tx, y) \leq_{\mathcal{C}(P\backslash\{0\})} 0, \quad \forall y \in C$$

and so $-(Tx, y) \in \mathcal{C}(P\backslash\{0\})$ for all $y \in C$. Since $P = \cup_{i\in I} P_i$, it follows that $-(Tx, y) \in \mathcal{C}(P_i\backslash\{0\})$ for all $y \in C$ and $i \in I$. Thus, $x \in S^{P_i}_{\mathrm{MVCP}}$ and so $x \in \cap_{i\in I} S^{P_i}_{\mathrm{MVCP}}$. Conversely, suppose that $x \in \cap_{i\in I} S^{P_i}_{\mathrm{MVCP}}$. Then $(Tx, x) = 0$ and

$$(Tx, y) \leq_{\mathcal{C}(P_i\backslash\{0\})} 0, \quad \forall y \in C, \ i \in I.$$

This implies that $-(Tx, y) \in \mathcal{C}(P_i\backslash\{0\})$ for all $y \in C$ and $i \in I$, and so $-(Tx, y) \in \mathcal{C}(P\backslash\{0\})$ for all $y \in C$. It follows that $x \in S^{P}_{\mathrm{MVCP}}$. This completes the proof. □

Example 2.6 Let $X = Y = R^2$, $C = [0, +\infty) \times [0, +\infty)$, and $P = P_1 \cup P_2$, where

$$P_1 = \left\{(x, y) : x \geq 0, 0 \leq y \leq \frac{x}{2}\right\}, \quad P_2 = \left\{(x, y) : y \geq 0, 0 \leq x \leq \frac{y}{2}\right\}.$$

Let T: $X \to L(X, Y)$ be defined by

$$Tx = \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix}, \quad \forall x = (x_1, x_2) \in X.$$

Then it is easy to see that $S^{P}_{\mathrm{SVCP}} = \{(0, 0)\}$. In fact, for each $x^* \in C$, $(T(x^*), x^*) = 0$ implies that $x^* = (0, 0)$ and

$$(Tx^*, y) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \geq_P 0, \quad \forall y = (y_1, y_2) \in C.$$

Similarly, we have $S^{P_i}_{\mathrm{SVCP}} = \{(0, 0)\}$ for $i = 1, 2$. Thus, $S^{P}_{\mathrm{SVCP}} = S^{P_1}_{\mathrm{SVCP}} \bigcup S^{P_2}_{\mathrm{SVCP}}$.

Example 2.7 Let X, Y, C, P, P_1 and P_2 be the same as in Example 2.6. Let T: $X \to L(X, Y)$ be defined by

$$Tx = \begin{pmatrix} 0 & 2 \\ 0 & x_2 \end{pmatrix}, \quad \forall x = (x_1, x_2) \in X.$$

Then

$$(Tx, x) = \begin{pmatrix} 0 & 2 \\ 0 & x_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2x_2 \\ x_2^2 \end{pmatrix}, \quad \forall x = (x_1, x_2) \in C.$$

It is easy to check that

$$S^{P}_{\mathrm{PVCP}} = [0, +\infty) \times \{0\} \cup [0, +\infty) \times \{1\} \cup [0, +\infty) \times \{4\}.$$

Similarly, we have

$$S^{P_1}_{\mathrm{PVCP}} = [0, +\infty) \times \{0\} \cup [0, +\infty) \times \{1\} \cup [0, +\infty) \times [4, +\infty)$$

and

$$S^{P_2}_{\mathrm{PVCP}} = [0, +\infty) \times [0, 1] \cup [0, +\infty) \times \{4\}.$$

Thus, $S^{P}_{\mathrm{PVCP}} = S^{P_1}_{\mathrm{PVCP}} \cap S^{P_2}_{\mathrm{PVCP}}$.

Example 2.8 Let X, Y, C, P, P_1, P_2 and T be the same as in Example 2.6. Then it is easy to see that $S^P_{\mathrm{MVCP}} = \{(0,0)\}$ and $S^{P_i}_{\mathrm{MVCP}} = \{(0,0)\}$ for $i = 1,2$. Thus, $S^P_{\mathrm{MVCP}} = S^{P_1}_{\mathrm{MVCP}} \cap S^{P_2}_{\mathrm{MVCP}}$.

Similarly, we have the following results.

Theorem 2.9 *Let $T: X \to L(X,Y)$ and $P = \bigcup_{i\in I} P_i$, where P_i is a closed, pointed, and convex cone in Y for $i \in I$. Then*

$$S^P_{\mathrm{SVVI}} \supset \bigcup_{i\in I} S^{P_i}_{\mathrm{SVVI}}, \quad S^P_{\mathrm{SMVVI}} \supset \bigcup_{i\in I} S^{P_i}_{\mathrm{SMVVI}}, \quad S^P_{\mathrm{MVVI}} = \bigcap_{i\in I} S^{P_i}_{\mathrm{MVVI}},$$

where $S^{P_i}_{\mathrm{SVVI}}$, $S^{P_i}_{\mathrm{SMVVI}}$, and $S^{P_i}_{\mathrm{MVVI}}$ are respectively the solution sets of the following problems: find $x \in C$ such that

$$(Tx, y - x) \geq_{P_i} 0, \quad \forall y \in C,$$
$$(Ty, y - x) \geq_{P_i} 0, \quad \forall y \in C$$

and

$$(Tx, y - x) \leq_{C(P_i \setminus \{0\})} 0, \quad \forall y \in C.$$

We now consider the minimal element problem.

Theorem 2.10 *Suppose that $P = \bigcup_{i\in I} P_i$, where P_i is a closed, pointed, and convex cone in Y for $i \in I$. Then, for any subset $A \subset Y$,*

$$Min_P A = \bigcup_{i\in I} Min_{P_i} A, \quad Min_{C(P\setminus\{0\})} A = \bigcup_{i\in I} Min_{C(P_i\setminus\{0\})} A.$$

Proof It is easy to see that

$$\begin{aligned} x \in \mathrm{Min}_P A &\Longleftrightarrow x \leq_P y, \quad \forall y \in A \\ &\Longleftrightarrow \exists\, i \in I \text{ such that } x \leq_{P_i} y, \quad \forall y \in A \\ &\Longleftrightarrow x \in \bigcup_{i\in I} \mathrm{Min}_{P_i} A. \end{aligned}$$

Thus,

$$\mathrm{Min}_P A = \bigcup_{i\in I} \mathrm{Min}_{P_i} A.$$

The second equality follows from the first equality directly. This completes that proof. □

3 Weak vector optimization problems

In this section, we give some results concerned with relations of solution sets for weak vector complementarity problems, weak vector variational inequalities, and weak vector optimization problems.

Let (X, C) be an ordered Banach space with $intC \neq \emptyset$, Y be a Banach space, P be a closed and pointed cone in Y with $intP \neq \emptyset$, and $T: X \to L(X,Y)$. We consider the *Weak Vector Complementarity Problem* (WVCP): finding $x \in C$, such that

$$(Tx, x) \geq_{C(intP)} 0, \quad (Tx, y) \leq_{C(intP)} 0, \quad \forall y \in C.$$

We also consider the *Weak Vector Variational Inequality* (WVVI): finding $x \in C$, such that

$$(Tx, y - x) \leq_{\mathcal{C}(intP)} 0, \quad \forall y \in C.$$

We denote by S^P_{WVCP} and S^P_{WVVI} the solution sets of WVCP and WVVI, respectively.

Let A be a nonempty subset of Y. We say that $a \in A$ is a weakly minimal point of the set A with respect to P if $y \leq_{\mathcal{C}(intP)} a$ for all $y \in A$. The set of all weakly minimal points of A is denoted by $\text{Min}_{\mathcal{C}(intP)}A$.

Let $T : X \to L(X, Y)$ be a mapping. Define the feasible set associated with T as follows:

$$\mathcal{F}_w = \{x \in X : \ x \in C, \quad (Tx, y) \leq_{\mathcal{C}(intP)} 0, \quad \forall y \in C\}.$$

Let $f(x) = (Tx, x)$ for all $x \in C$. We now consider the *Weak Vector Optimization Problem* (WVOP):

$$\text{Min}_{\mathcal{C}(intP)} f(x) \qquad \text{subject to} \quad x \in \mathcal{F}_w.$$

A point x is called a weakly minimal solution of WVOP if $f(x)$ is a weakly minimal point of WVOP, i.e., $f(x) \in \text{Min}_{\mathcal{C}(intP)} f(\mathcal{F}_w)$. We denote the set of all weakly minimal solutions of WVOP by E_w, and the set of all weakly minimal points of WVOP by H_w. Then $f(E_w) = H_w$.

Theorem 3.1 *Suppose that $f(E_w) \neq \emptyset$. If there exists $x \in E_w$ such that $f(x) \geq_{\mathcal{C}(intP)} 0$, then the WVCP is solvable.*

Proof Let $x \in E_w$ and $f(x) \geq_{\mathcal{C}(intP)} 0$. Then $x \in C$ and

$$(Tx, x) = f(x) \geq_{\mathcal{C}(intP)} 0, \quad (Tx, y) \leq_{\mathcal{C}(intP)} 0, \quad \forall y \in C.$$

It follows that x is a solution of WVCP. This completes the proof. □

We now consider the following problems.

The WVOP$_l$: for a given $l \in L(X, Y)$, finding $x \in \mathcal{F}_w$ such that $l(x) \in \text{Min}_{\mathcal{C}(intP)} l(\mathcal{F}_w)$;

The weak minimal element problem (WMEP): finding $x \in \mathcal{F}_w$ such that $x \in \text{Min}_{\mathcal{C}C}\mathcal{F}_w$.

The weak vector unilateral optimization problem (WVUOP): finding $x \in C$ such that $f(x) \in \text{Min}_{\mathcal{C}(intP)} f(C)$.

Theorem 3.2 *Let $T = Df$ be the Frechet derivative of an operator f: $X \to Y$. Then x solves (WVUOP) implies that x solves (WVVI).*

Proof Let x be a solution of WVUOP. Then $x \in C$ and $f(x) \in \text{Min}_{\mathcal{C}(intP)} f(C)$, i.e., $f(y) \leq_{\mathcal{C}P} f(x)$ for all $y \in C$. Since, C is a convex cone,

$$f(x) \geq_{\mathcal{C}(intP)} f(x + t(w - x)), \quad 0 < t < 1, \ w \in C.$$

It follows that

$$\frac{1}{t}[f(x + t(w - x)) - f(x)] \leq_{\mathcal{C}(intP)} 0.$$

Since, f is Frechet differentiable on X and $\mathcal{C}(intP)$ is closed, letting $t \to 0^+$, we get

$$(Df(x), w - x) \leq_{\mathcal{C}(intP)} 0, \quad \forall w \in C,$$

which is WVVI. This completes the proof. □

Theorem 3.3 *Let l be a linear operator. If l is positive with respect to $\mathcal{C}C$ and $\mathcal{C}(intP)$, then x solves $(WMEP)$ implies x solves $(WVOP)_l$.*

Proof Let x be a solution of (WMEP). Then $x \in \mathcal{F}_w$ and $y \leq_{\mathcal{C}C} x$ for all $y \in \mathcal{F}_w$, where

$$\mathcal{F}_w = \{x \in X : \ x \in C, \ (Tx, y) \leq_{\mathcal{C}(intP)} 0, \quad \forall y \in C\}.$$

For any $z \in \mathcal{F}_w$, we know that $z \leq_{\mathcal{C}C} x$. Since l is positive with respect to $\mathcal{C}C$ and $\mathcal{C}(intP)$, it follows that $l(z) \leq_{\mathcal{C}(intP)} l(x)$ for all $z \in \mathcal{F}_w$. So x solves $(\text{WVOP})_l$. This completes the proof.

Next we assume that P is a union of some convex cones.

Theorem 3.4 *Let $T : X \to L(X, Y)$ and $P = \bigcup_{i \in I} P_i$, where P_i is a closed, pointed, and convex cone in Y for $i \in I$. Then*

$$S^P_{\text{WVCP}} \subset \bigcap_{i \in I} S^{P_i}_{\text{WVCP}} \quad \textit{and} \quad S^P_{\text{WVVI}} \subset \bigcap_{i \in I} S^{P_i}_{\text{WVVI}},$$

where $S^{P_i}_{\text{WVCP}}$ $(i \in I)$ is the solution set of the following vector complementarity problem: find $x \in C$ such that

$$(Tx, x) \geq_{\mathcal{C}(intP_i)} 0, \quad (Tx, y) \leq_{\mathcal{C}(intP_i)} 0, \quad \forall y \in C$$

and $S^{P_i}_{\text{WVVI}}$ is the solution set of the following problem: find $x \in C$ such that

$$(Tx, y - x) \leq_{\mathcal{C}(intP_i)} 0, \quad \forall y \in C.$$

Proof Let $x \in S^P_{\text{WVCP}}$. Then $x \in C$ and

$$(Tx, x) \geq_{\mathcal{C}(intP)} 0, \quad (Tx, y) \leq_{\mathcal{C}(intP)} 0, \quad \forall y \in C.$$

Since $\cup_{i \in I} intP_i \subset intP$,

$$(Tx, x) \in \mathcal{C}(intP_i), \quad (Tx, y) \in -\mathcal{C}(intP_i), \quad \forall y \in C, \ i \in I.$$

It follows that $x \in S^{P_i}_{\text{WVCP}}$ for all $i \in I$. Similarly, we can prove that $S^P_{\text{WVVI}} \subset \bigcap_{i \in I} S^{P_i}_{\text{WVVI}}$. This completes the proof. □

Theorem 3.5 *Suppose that $P = \bigcup_{i \in I} P_i$, where P_i is a closed, pointed, and convex cone in Y for $i \in I$. Then, for any subset $A \subset Y$,*

$$Min_{\mathcal{C}P} A \subset \bigcap_{i \in I} Min_{\mathcal{C}(intP_i)} A.$$

Proof Let $x \in \text{Min}_{\mathcal{C}(intP)} A$. Then $y \leq_{\mathcal{C}(intP)} x$ for all $y \in A$ and so $x - y \in \mathcal{C}(intP)$ for all $y \in A$. Since $\cup_{i \in I} intP_i \subset intP$, it follows that

$$x - y \in \mathcal{C}(intP_i), \quad \forall y \in A, \ i \in I.$$

Thus,

$$y \leq_{\mathcal{C}(intP_i)} x, \quad \forall y \in A, \ i \in I$$

and so $x \in \cap_{i \in I} \text{Min}_{\mathcal{C}(intP_i)} A$. This completes that proof. □

4 Conclusions

The main result obtained in the paper is that, for a number of vector optimization problems where the ordering relation is defined by the union of a number of convex cones, the solution set of the problem concerned is shown to be the intersection of the solution sets of all vector optimization subproblems which are defined by each convex cone. This result looks interesting and may be useful in the design of optimal algorithms to find the whole solution set of vector optimization problems with a nonconvex ordering cone.

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References

1. Ansari, Q.H., Lai, T.C., Yao, J.C.: On the equivalence of extended generalized complementarity and generalized least-element problems. J. Optim. Theory Appl. **102**, 277–288 (1999)
2. Chen, G.Y., Huang, X.X., Yang, X.Q. *Vector Optimization: Set-valued and Variational Analysis*, Lecture Notes in Economics and Mathematical Systems 541, Springer-Verlag, Berlin (2005)
3. Chen, G.Y., Yang, X.Q.: The vector complementarity problem and its equivalences with the weak minimal element in ordered Banach spaces. J. Math. Anal. Appl. **153**, 136–158 (1990)
4. Cryer, C.W., Dempster, A.H.: Equivalence of linear complementarity problems and linear program in vector lattice Hilbert space. SIAM J. Control. Optim. **18**, 76–90 (1980)
5. Giannessi, F. : Theorems of alternative, quadratic programs and complementarity problems. In: Cottle, R.W., Giannessi, F., Lions, J.L. (eds.) Variational Inequalities and Complementarity Problems, pp. 151–186. Wiley, New York (1980)
6. Giannessi, F. (ed.): Vector Variational Inequalities and Vector Equilibrium. Kluwer Academic Publishers, Dordrecht, Boston, London (2000)
7. Göpfert, A., Riahi, H., Tammer, C., Zălinescu, C.: Variational Methods in Partially Ordered Spaces. Springer-Verlag, New York (2003)
8. Huang, N.J., Fang, Y.P.: Strong vector F-complementary problem and least element problem of feasible set. Nonlinear Anal. **61**, 901–918 (2005)
9. Huang, N.J., Yang, X.Q., Chan, W.K. Vector complementarity problems with a variable ordering relation. *Eur. J. Oper. Res.* (in press)
10. Isac, G., Bulavsky, V.A., Kalashnikov, V.V.: Complementarity, Equilibrium, Efficiency and Economics. Kluwer Academic Publishers, Dordrecht, Boston, London (2002)
11. Riddell, R.C.: Equivalence of nonlinear complementarity problems and least element problems in Banach lattice. Math. Oper. Res. **6**, 462–474 (1981)
12. Rubinov, A.M.: Abstract Convexity and Global Optimization. Kluwer Academic Publishers, Dordrecht, Boston, London (2000)
13. Rubinov, A.M., Gasimov, R.N.: Scalarization and nonlinear scalar duality for vector optimization with preferences that are not necessarily a pre-order relation. J. Glob. Optim. **29**, 455–477 (2004)
14. Yang, X.Q.: Vector complementarity and minimal element problems. J. Optim. Theory Appl. **77**, 483–495 (1993)